General Moment Model of Beam Transport*

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Abstract

Using the Hamiltonian structure of the Vlasov equation, we develop a general, relativistic, three-dimensional model of beam transport based on phase space moments of the beam particle distribution. Evolution equations for these moments are derived from the non-canonical Poisson bracket for the Vlasov equation. In this model, the beam centroid experiences the full non-linear forces in the system while the higher order moments are coupled to both the centroid and to various spatial derivatives of the applied fields. For example, when moments up to second order are retained, the physics content is similar to considering linearized forces. Given the large number of equations (there are 27 equations when all second order moments are kept) and their algebraic complexity, the use of symbolic computation in the derivation was critical to ensuring the correctness of the equations. This approach also allows for analytical verification of conservation laws associated with the model. The initial investigations [1] have considered only externally applied fields, however in principle space-charge forces can also be included. We discuss the necessary extensions to the basic theory needed to model ionization cooling for the muon collider[2].

1 INTRODUCTION

Using moments to construct reduced models of phase space dynamics is not a new idea; for example, see Refs. [3, 4] for a linac application and Ref. [5] for general approach to Hamiltonian field theories. The desire such models is clear. Tracking individual particles is computationally very intensive (if reasonable statistics are to be obtained) and in many instances the detailed information that tracking produces is not of great interest. Furthermore, often the beam dynamics is largely linear so representing the bulk of the beam by particles is inefficient. One-dimensional moment equations are of significant pedagogical interest as they provide a simple means for understanding a variety of elementary beam dynamics. While such models are of little use in detailed studies of beam transport, there still exists the possibility of extracting much of the beam behaviour without resorting to tracking individual particles.

Here we present a formalism for a general, fully relativistic, three-dimensional moment description of beam transport based upon the noncanonical Hamiltonian structure of the Vlasov-Poisson equation [6]. This approach has many attractive features. By using a formulation based on Poisson brackets, derivation of the equations of motion for the moments is purely mechanical and is ideally suited to the application of symbolic manipulation. In this approach, one approximates the Hamiltonian and bracket in terms of moments and then uses Hamilton's equations to obtain equations of motion. Not only is this procedure less cumbersome than directly averaging the single particle equations, it also eliminates the difficulty of determining (in a more less *ad hoc* fashion) a consistent ordering of the moment expansion. As is common in this type of reduction, one finds that even when the external forces are linearized, for the model to conserve energy (and typically other invariants also, if they exist) it is necessary that the evolution equations retain various terms that are nonlinear in the moments. When the equations are derived from a Poisson bracket and Hamiltonian these nonlinear terms automatically appear as needed. Moment representations are intrinsically statistical in nature and are not susceptible to (nor sensitive to) noise associated with finite particle affects. Terms in the moments equations have two origins: kinematic terms (*i.e.,* those terms associated with the free-streaming of phase space) and terms associated with electromagnetic forces. It turns out the that the kinematic terms have the form of an expansion in the reciprocal of the centroid γ-factor while the electromagnetic forces are essentially Taylor expanded about the centroid location to an order that depends on the order of the moments being retained. Typically one finds that the moment equations do not close, *i.e.,* the equations of motion for a set of moments of a particular order tend to include couplings to moments of higher order. There are numerous methods for imposing a closure. The observation that the moment model is a combination of an asymptotic expansion in γ^{-1} and a Taylor expansion of the applied forces means that for even mildly relativistic beams ($\gamma \sim 2$) and for forces that do not vary to drastically over the extent of the beam, simple truncation is a reasonably accurate closure. It is also the case that imposing a closure may well destroy the Hamiltonian structure of the moment system (in the sense that the bracket typically no longer satisfies the Jacobi identity). While philosophically one might prefer an approximation that fully retains the Hamiltonian character of the underlying dynamics, this loss does not lessen the power of the bracket approach to deriving equations of motion.

2 HAMILTONIAN FORMULATION OF THE VLASOV EQUATION

We examine the simplest case where we ignore the self-interactions of the beam (*i.e.* we ignore space charge ef-

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fects), although more general models are possible. In this case, the beam dynamics are governed by the relativistic Vlasov equation with *external* fields:

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f + \frac{q}{m}\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\boldsymbol{\nabla}_p f = 0\,, \qquad (1)$$

where f is the phase-space particle density, q and m are the particle mass and charge, respectively and $\boldsymbol{\nabla}_p = \partial/\partial\mathbf{p}$. Writing the particle phase-space distribution function in terms of the *canonical* momentum, $\mathfrak{p}$, the relativistic Vlasov equation can be written as [6]:

$$\frac{\partial f}{\partial t} = \{\, f\,, H\,\} \qquad (2)$$

where the Hamiltonian H is given by:

$$H = \int d^3\mathbf{r}\, d^3\mathfrak{p}\,\left(\gamma\, m\, c^2 + \phi\right) f(\mathbf{r},\mathfrak{p})\,. \qquad (3)$$

The (noncanonical) Poisson bracket $\{\,\cdot\,,\cdot\,\}$ of any functionals F and G of f is given by

$$\{\, F, G\,\} = \int d^3\mathbf{r}\, d^3\mathfrak{p}\, f\left[\frac{\delta F}{\delta f}\,,\frac{\delta G}{\delta f}\right] \qquad (4)$$

where $[\,\cdot\,,\cdot\,]$ is the usual phase-space Poisson bracket:

$$[\, F\,, G\,] = \boldsymbol{\nabla} F\cdot\boldsymbol{\nabla}_{\mathfrak{p}}\, G - \boldsymbol{\nabla} G\cdot\boldsymbol{\nabla}_{\mathfrak{p}}\, F\,. \qquad (5)$$

For the purposes of studying charged-particle optics, it is more convenient to express f in terms of the mechanical momentum, $\mathbf{p} = \mathfrak{p} - q/c\mathbf{A}$, where $\mathbf{A}$ is the vector potential. Under this coordinate transformation

$$\frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t} - \frac{q}{c}\,\frac{\partial\mathbf{A}}{\partial t}\cdot\boldsymbol{\nabla}_p\,, \qquad (6)$$

$$\boldsymbol{\nabla}_{\mathfrak{p}} \longrightarrow \boldsymbol{\nabla}_p\,, \qquad (7)$$

$$\boldsymbol{\nabla} \longrightarrow \boldsymbol{\nabla} - \frac{q}{c}\,\boldsymbol{\nabla} A_k\frac{\partial}{\partial p_k}. \qquad (8)$$

Applying this change of variables to the brackets gives,

$$\begin{aligned}\{\, F, G\,\} \;=\; &\int d^3\mathbf{r}\, d^3\mathbf{p}\; f\left[\frac{\delta F}{\delta f}\,,\frac{\delta G}{\delta f}\right]_{xp} \\ &+\frac{q}{c}\int d^3\mathbf{r}\, d^3\mathbf{p}\; f\,\mathbf{B}\cdot\boldsymbol{\nabla}_p\frac{\delta F}{\delta f}\times\boldsymbol{\nabla}_p\frac{\delta G}{\delta f}, \qquad (9)\end{aligned}$$

where $[\,\cdot\,,\cdot\,]_{xp}$ is given by (5) with $\mathfrak{p}$ replaced by $\mathbf{p}$ and we will subsequently suppress this subscript. The Vlasov equation now reads

$$\frac{\partial f}{\partial t} - \frac{q}{c}\,\frac{\partial\mathbf{A}}{\partial t}\cdot\boldsymbol{\nabla}_p f = \{\, f\,, H\,\} \qquad (10)$$

Upon close examination of (10), we see that $-(1/c)\partial\mathbf{A}/\partial t$ enters the expression for $\partial f/\partial t$ in exactly the same way as does $\boldsymbol{\nabla}\phi$. The following trick is useful (but not necessary): make the formal identification[1] $\boldsymbol{\nabla}\varphi = \boldsymbol{\nabla}\phi + (1/c)\partial\mathbf{A}/\partial t$, and write (10) as

$$\frac{\partial f}{\partial t} = \left\{\, f\,, \widetilde{H}\,\right\}, \qquad (11)$$

where $\widetilde{H}$ is obtained from H by the replacement $\phi\to\varphi$. The chain rule can then be used to obtain the time derivative of any functional of f:

$$\begin{aligned}\frac{dF}{dt} &= \int d^3\mathbf{r}\, d^3\mathbf{p}\,\frac{\delta F}{\delta f}\left\{\, f\,, \widetilde{H}\,\right\} + \frac{\partial F}{\partial t} \\ &= \left\{\, F\,, \widetilde{H}\,\right\} + \frac{\partial F}{\partial t}. \qquad (12)\end{aligned}$$

3 MOMENT EQUATIONS

Our moment models are based on an expansion of the phase space coordinates about the location of the beam centroid:

$$\mathfrak{z}_j = \langle\mathfrak{z}_j\rangle + \delta\mathfrak{z}_j\,, \qquad (13)$$

where $\{\mathfrak{z}_j\}_{j=1}^6 = \{x, y, z, p_x, p_y, p_z\}$ and $\langle\cdot\rangle$ is the normalized phase-space average. To simplify the presentation, we keep moments only up to second order, but it is clear that these procedures can be carried out to arbitrary order. We define

$$M_i \equiv \langle\mathfrak{z}_i\rangle \quad\text{and}\quad M_{ij} \equiv \langle\delta\mathfrak{z}_i\,\delta\mathfrak{z}_j\rangle\,. \qquad (14)$$

The first order moments, M_i, represent the beam centroid, while the second-order moments, M_{ij}, represent the phase-space extent of the beam.

Substituting (13) into the expression for $\widetilde{H}$ and keeping terms through second order in $\delta\mathfrak{z}_i$, we obtain a Hamiltonian for the moment system:

$$\begin{aligned}\widehat{H} \;=\; & f_0\left(\gamma_0\, m\, c^2 + q\,\varphi\right) - f_0\,\frac{m\, c^2}{2\gamma_0^3}\Big(\langle\delta p_x^2\rangle\,\langle p_x\rangle^2 \\ &+ \langle\delta p_y^2\rangle\,\langle p_y\rangle^2 + \langle\delta p_z^2\rangle\,\langle p_z\rangle^2 + 2\langle\delta p_x\,\delta p_y\rangle\,\langle p_x\rangle\,\langle p_y\rangle \\ &+ 2\langle\delta p_x\,\delta p_z\rangle\,\langle p_x\rangle\,\langle p_z\rangle + 2\langle\delta p_y\,\delta p_z\rangle\,\langle p_y\rangle\,\langle p_z\rangle\Big) \\ &+ f_0\,\frac{m\,c^2}{2\gamma_0}\,\left(\langle\delta p_x^2\rangle + \langle\delta p_y^2\rangle + \langle\delta p_z^2\rangle\right) \\ &- f_0\, q\left(\frac{\langle\delta x^2\rangle}{2}\,\frac{\partial E_x}{\partial x} + \frac{\langle\delta y^2\rangle}{2}\,\frac{\partial E_y}{\partial y} + \frac{\langle\delta z^2\rangle}{2}\,\frac{\partial E_z}{\partial z}\right. \\ &\left. + \langle\delta x\,\delta y\rangle\,\frac{\partial E_x}{\partial y} + \langle\delta x\,\delta z\rangle\,\frac{\partial E_x}{\partial z} + \langle\delta y\,\delta z\rangle\,\frac{\partial E_y}{\partial z}\right), \qquad (15)\end{aligned}$$

where $\gamma_0 = \left[1 + (\langle p_x\rangle^2 + \langle p_y\rangle^2 + \langle p_z\rangle^2)/(m^2\, c^2)\right]^{1/2}$, $f_0 = \int d^3\mathbf{r}\, d^3\mathbf{p}\; f$, and all external fields are evaluated at the centroid position.

Since moments are clearly functionals of f, (12) and (9) give the necessary equations of motion. In evaluating (9),

[1] In general this is equation is not solvable for φ; the solvability condition is $\partial\mathbf{B}/\partial t = 0$.

note that for a functional of f that can be written as a *function* of the moments alone, *i.e.*, $F[f] = \widehat{F}(M_i, M_{ij})$, we have

$$\begin{aligned} \frac{\delta F}{\delta f} &= \frac{\partial \widehat{F}}{\partial M_i}\,\frac{\delta M_i}{\delta f} + \frac{\partial \widehat{F}}{\partial M_{ij}}\,\frac{\delta M_{ij}}{\delta f} \\ &= \frac{\partial \widehat{F}}{\partial M_i}\,\frac{1}{f_0}\,\mathfrak{z}_i + \frac{\partial \widehat{F}}{\partial M_{ij}}\,\frac{1}{f_0}\,\mathfrak{z}_i\,\mathfrak{z}_j . \end{aligned} \tag{16}$$

Combining the above we can write the equations of motion for the moments:

$$\dot{M}_i = \left\{ M_i \,, \widehat{H} \right\} \quad \text{and} \quad \dot{M}_{ij} = \left\{ M_{ij} \,, \widehat{H} \right\}. \tag{17}$$

While evaluating (17) is conceptually straightforward, these equations are both numerous (there a total of 27 moments through second order) and algebraically very complex. To overcome this complexity and to ensure the correctness of the resulting equations, we have implemented (9), (12), and (16) using symbolic algebra. In addition to deriving the equations, our symbolic algebra program also produces the necessary source code for the numerical implementation of these equations. This approach not only increases our confidence that the final simulation code is correct but also enables symbolic identification and verification of a variety of conserved quantities (which are then used as diagnostics in the simulation). Examples of such invariants include, energy, six-dimensional emittance ($\det M_{ij}$), angular momentum (in axially symmetric systems), and longitudinal and transverse emittance separately, in systems where the dynamics decouples.

Note that, exactly one expects, the first two terms in $\widehat{H}$ are the Hamiltonian for a single particle, whose trajectory is that of the beam centroid. As higher order moments are added to the model, this "particle" acquires internal structure (in 6-d phase space) which results in additional terms in the equations of motion of the first order moments.

4 APPLICATION TO MUON COOLING

To study the ionization cooling of muons we must extend our formulation to include the non-Hamiltonian interaction of the beam with material absorbers. This interaction can be divided into inelastic (ionization energy loss) and elastic ("multiple scattering") parts. The inelastic part can be viewed as a frictional force in the direction of the beam momentum and the contributions to the moment equations can be found by taking appropriate averages of this force term. The elastic piece is due to small angle scattering in random directions, and thus can only be included in a statistically averaged sense. We have developed a simulation code that takes these effects into account for studying a section of the transverse cooling channel [7] of the muon collider. Our preliminary results [2] show that the moment model is in very good agreement with particle tracking studies and the simulation promises to be a useful design tool.

5 CONCLUSIONS

We have presented a formalism for constructing models of beam dynamics based upon moments of arbitrary order. It is also possible to construct "semi-discrete" models; for example, by averaging only over the transverse phase space, one obtains a system where the transverse dynamics are determined by moments while a full kinetic description is retained longitudinally. One can also envision constructing a δf method using a moment formalism to represent the core of a beam. Such a method could yield important kinetic information (say for studies of halo formation) without the computational burden of using particles to model the beam core. As an application of moment methods, we have studied an ionization cooling channel for the muon collider and have found close agreement between our simulation and full particle tracking, verifying the utility of moments models as design tools. Here we have closed the system by discarding higher-order moments, however, other closures are equally justified: for example, one can, by ignoring correlations, approximate the higher moments using products of lower order moments. Understanding how the choice of closure affects the accuracy of the model will be the subject of future work.

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